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# The stability of spatially periodic flows

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The stability characteristics for spatially periodic parallel flows of an incompressible fluid (both inviscid and viscous) are studied. A general formula for the determination of the stability characteristics of periodic flows to long waves is obtained, and applied to approximate numerically the stability curves for the sinusoidal velocity profile. The neutral curve for the sinusoidal velocity profile is obtained analytically. The stability of two broken-line velocity profiles in an inviscid fluid is studied and the results are used to describe the overall pattern for the sinusoidal velocity profile in the case of long waves. In an inviscid fluid it is found that all periodic flows (other than the trivial flow in which the basic velocity is constant) are unstable to long waves with a value of the phase speed determined by simple integrals of the basic flow. In a viscous fluid it is found that the sinusoidal velocity profile is very unstable with the inviscid solution being a good approximation to the solution of the viscous problem when the value of the Reynolds number is greater than about 20.

## 1. Introduction

Over the last century the stability of parallel flows of incompressible fluids (both inviscid and viscious) has been extensively studied. The techniques used usually involve examination of the growth of an infinitesimal wave-like perturbation of a basic flow of the form $\mathbf{U} = U(z)\mathbf{i}$. Squire (1933) has shown that the three-dimensional problem can be reduced to an equivalent two-dimensional problem since the most unstable disturbance is two-dimensional. Previously, most of the work on unbounded velocity profiles has been concerned with the stability of profiles which are uniform as $z \rightarrow \pm\infty$. In this paper we shall consider flows with $U(z) = U(z+\lambda)$ for some period $\lambda$ and all $z$. In addition to interest in the fundamental phenomena of instability of periodic flows there is also interest in various applications. Gill (1974), while studying the stability of a finite-amplitude Rossby wave, noted that in a particular limit his problem reduced to that of the stability of a plane parallel periodic flow. It appears that the same result will arise for many other kinds of waves as a consequence of an analogous limit (see Drazin 1977). Our results show how finite-amplitude waves in an inviscid fluid in this limit are linearly unstable. Green (1974) also considered the stability of a sinusoidally varying velocity profile in connection with some work on two-dimensional turbulence. Although the main applications of periodic velocity profiles are naturally concerned with waves, periodic flows will also arise behind any periodic structure such as a regularly spaced grid.

The most obvious periodic profile to consider is the sinusoidal velocity profile. It is well known that this profile becomes stable to all wave-like perturbations when the boundary separation is less than $\pi$ (see, for example, Drazin & Howard 1966). One might also remark on the general stability characteristics for periodic flows. For an inviscid fluid many useful results have been developed for flows with zero normal velocity on the boundaries (see, for example, Drazin & Howard 1966). For an unbounded periodic velocity profile it can be shown that the disturbance amplitude function is either periodic or bounded according as a certain parameter $\gamma_I$ is rational or irrational and these general results still hold for the periodic solutions. For $\gamma_I$ irrational we can tentatively appeal to continuity to obtain the same results. Probably the most important of these results is the semicircle theorem (Howard 1961), which states that for unstable solutions the phase speed $c$ lies in the semicircle in the complex plane defined by

$$\{c_R - \tfrac{1}{2}(e+f)\}^2 + c_I^2 \leqslant \{\tfrac{1}{2}(e-f)\}^2,$$

where $e$ is the maximum and $f$ the minimum value of $U(z)$.

It is also interesting to speculate on the existence of a critical stabilizing Reynolds number $R_c$. For each unbounded flow of jet type, i.e. with $U(\infty) = U(-\infty)$ there exists a positive $R_c$ (Howard 1959) but each unbounded flow of shear-layer type, i.e. with $U(\infty) \neq U(-\infty)$, has $R_c = 0$ (Tatsumi & Gotch 1960). It remains an open question as to whether periodic flows have a positive value of $R_c$ but, for the $\mathrm{sn}\,(z,m)$ profile, as the modulus $m$ of this Jacobian elliptic function increases towards 1, the value of $R_c$ tends to zero because the profile tends to the hyperbolic-tangent shear layer. Therefore the critical Reynolds number may be arbitrarily small for some period flows. This conclusion is supported by the apparent inability of Synge's (1938) method to give a non-zero critical Reynolds number for a general periodic flow. At any rate, it seems likely that any critical Reynolds number must depend on the period of the basic flow.

In §2 we present the equations of linear stability theory for plane parallel flows together with a brief summary of the relevant ideas of Floquet theory which are used throughout the rest of the paper. In §3 we derive, in the case of an inviscid fluid, a formula for the complex phase speed for small values of the wavenumber. This is followed in §4 by derivation of the exact analytic solutions of the inviscid flow problem for two broken-line velocity profiles. These exact results are then compared with those calculated from the small-wavenumber formula developed in §3. In §5 we present numerical and analytic results for the inviscid flow problem with a sinusoidal velocity profile, and then, in §6, we incorporate viscosity into this problem. Finally, in §7, the results presented in the previous sections are discussed and compared.

## 2. Governing equations and Floquet theory

Suppose that the basic flow with velocity $U^*(z^*)$ in the $x^*$ direction is dimensionally characterized by some length scale $L$ and velocity scale $V$; then we shall choose dimensionless variables as follows: $x = x^*/L$, $t = t^*V/L$ and $U(z) = U^*(z^*)/V$. We can further define a Reynolds number as $R = VL/\nu$ where $\nu$ is the kinematic viscosity of the fluid. We then assume that the streamfunction for the perturbation of the basic flow is of the form $\psi = \phi(z)\exp\{i\alpha(x-ct)\}$ in dimensionless form. Here $c = c_R + ic_I$ is a a complex wave velocity and $\alpha$ is a positive wavenumber. Then the sign of $c_I$ determines the stability of the basic flow to this perturbation. The linearized equations of motion now lead (cf. Lin 1955) to the Orr–Sommerfeld equation

$$(U-c)(\phi''-\alpha^2\phi)-U''\phi = \frac{1}{i\alpha R}\{\phi^{\mathrm{iv}}-2\alpha^2\phi''+\alpha^4\phi\}, \tag{1}$$

where primes denote differentiation with respect to $z$. For the case of an inviscid fluid (1) reduces to the Rayleigh equation

$$(U-c)(\phi''-\alpha^2\phi)-U'' = 0. \tag{2}$$

Here we are making the usual approximation of nearly parallel flow, assuming that $U(z)\,\mathbf{i}$ is an exact solution of the Navier–Stokes equation to obtain (1), but then taking a periodic $U(z)$ which is only an approximate solution of the Navier–Stokes equations.

For the types of problem under consideration where the basic flow $U(z)$ has period $\lambda$, both (1) and (2) are examples of a Floquet system. In this section we give the essential details of the methods of Floquet theory for the Orr–Sommerfeld equation (the modifications for dealing with the Rayleigh equation are obvious) and the reader is referred to Jordan & Smith (1977) for the general techniques of Floquet theory. We define $\mathbf{\Phi}(z)$ to be the $4\times 4$ fundamental matrix of the system (1)

$$d\mathbf{\Phi}(z)/dz = \mathbf{P}(z)\,\mathbf{\Phi}(z), \quad \mathbf{P}(z+\lambda) = \mathbf{P}(z), \tag{3}$$

whose elements are $\Phi_{ij} = d^{i-1}\phi_j/dz^{i-1}$ for $i,j = 1,2,3,4$ and are such that $\mathbf{\Phi}(0) = \mathbf{I}$, the identity matrix. The matrix of coefficients $\mathbf{P}(z)$ is therefore

$$\mathbf{P}(z) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ p_{41} & 0 & p_{43} & 0 \end{bmatrix}, \tag{4}$$

where $p_{41} = -i\alpha^3R(U-c)-i\alpha RU''-\alpha^4$ and $p_{43} = i\alpha R(U-c)+2\alpha^2$. Floquet's theorem then states that if the fundamental matrix $\mathbf{\Phi}(z)$ can be determined over any interval of the period of $z$ then it is determined for all $z$ since $\mathbf{\Phi}(z+\lambda) = \mathbf{\Phi}(z)\,\mathbf{E}$, where $\mathbf{E}$ is a constant non-singular matrix. Therefore, since we have chosen the fundamental matrix with $\mathbf{\Phi}(0) = \mathbf{I}$ we must have $\mathbf{E} = \mathbf{\Phi}(\lambda)$. It can then be shown that independent solutions of equation (1) may be written in the form

$$\phi = p(z)\exp\{\lambda^{-1}\log(\mu)z\}, \tag{5}$$

for each distinct eigenvalue $\mu$ of $\mathbf{E}$. The function $p(z)$ has period $\lambda$. When the eigenvalues are not all distinct, the coefficients corresponding to the $p(z)$ are more complicated. The constants $\mu$ are termed the characteristic multipliers of the Floquet system (3) and the corresponding characteristic exponents are determined by the relation $\gamma = \log(\mu)/\lambda$, where the principal value of the logarithm is taken. Therefore, in order to determine solutions which are stable in $z$ it is sufficient to compute the matrix $\mathbf{\Phi}(\lambda)$ and see whether any of its eigenvalues are such that $|\mu| = 1$. It should also be noted that

$$\mu_1\mu_2\mu_3\mu_4 = \det\mathbf{\Phi}(\lambda) = \det\mathbf{\Phi}(0)\exp\left\{\int_0^\lambda \mathrm{trace}\,\{\mathrm{P}(z)\}\,dz\right\} = 1, \tag{6}$$

which gives a useful check on the accuracy of the computations.

Equation (1) can not only be integrated directly but also solved by assuming a solution of the form

$$\phi = \exp(\gamma z) \sum_{n=-\infty}^{\infty} a_n \exp\{2\pi i n z/\lambda\}, \tag{7}$$

and then truncating the series at successively higher orders until convergence is plausibly obtained for the calculated values of $c$. This method is found to be more efficient than direct integration for large values of the Reynolds number.

In general we therefore have an eigenvalue relation of the form $F(c, \alpha^2, \alpha R, \gamma) = 0$, where $F$ is some integral function of $c$, $\alpha^2$, $\alpha R$ and $\gamma$. To solve the fluid dynamical problems, we are interested in the temporal stability of solutions which are bounded functions of $z$; so we write $c = c(\alpha^2, \alpha R, \gamma_I)$, where $\gamma_I$ is the imaginary part of $\gamma$.

We should also note that a solution (5) comprises a function of period $\lambda$ multiplied by an exponential factor. This means that if $\gamma_I\lambda/2\pi$ is rational the solution is periodic, whereas if $\gamma_I\lambda/2\pi$ is irrational the solution is bounded but not periodic.

## 3. The phase velocity for small values of the wavenumber

In this section we consider solutions to the Rayleigh equation (2) for small values of the wavenumber $\alpha$. We seek a solution of (2) in the form

$$\phi = \phi_0 + \alpha^2\phi_1 + \dots, \tag{8}$$

for small values of $\alpha$. Inserting (8) into (2) and equating powers of $\alpha^2$ we then obtain

$$(U-c)\phi_0'' - U''\phi_0, \quad \{(U-c)^2(\phi_n/(U-c))'\}' = (U-c)\phi_{n-1}, \quad n = 1, 2, \dots. \tag{9}$$

Clearly the zeroth-order equation has solution

$$\phi_0 = A(U-c) + B(U-c)\int^z (U-c)^{-2}\, dz_1, \tag{10}$$

for some constants $A$ and $B$. We now specify two sets of initial conditions at the origin:

$$\left.\begin{array}{l} (a) \text{ (denoted by subscript } a\text{);} \\ \qquad \phi_{0a}(0) = 1, \quad \phi_{na} = 0, \quad n \geqslant 1; \quad \phi'_{na}(0), \quad n \geqslant 0; \\ (b) \text{ (denoted by subscript } b\text{)} \\ \qquad \phi_{nb}(0) = 0, \quad n \geqslant 0; \quad \phi'_{0b}(0) = 1, \quad \phi'_{nb}(0) = 0, \quad n \geqslant 1. \end{array}\right\} \tag{11}$$

We can therefore obtain values for the constants $A_a$, $B_0$ (initial conditions ($a$)) and $A_b$, $B_b$ (initial conditions ($b$)). These values are then used to give the values of $\phi_0$ and $\phi'_0$ at $z = \lambda$:

$$\left.\begin{aligned} \phi_{0a}(\lambda) &= 1 - U'(0)\{U(0)-c\}\int_0^\lambda (U-c)^{-2}\, dz, \\ \phi'_{0a}(\lambda) &= -\{U'(0)\}^2\int_0^\lambda (U-c)^{-2}\, dz; \\ \phi_{0b}(\lambda) &= \{U(0)-c\}^2\int_0^\lambda (U-c)^{-2}\, dz, \\ \phi'_{0b}(\lambda) &= 1 + U'(0)\{U(0)-c\}\int_0^\lambda (U-c)^{-2}\, dz. \end{aligned}\right\} \tag{12}$$

Now, because we have a $2\times 2$ system, the eigenvalue problem for finding the characteristic multipliers $\mu$ reduces to

$$\mu^2-\{\phi_a(\lambda)+\phi'_b(\lambda)\}\mu+1 = 0. \tag{13}$$

Note that, in obtaining (13), we have used (6). This means that at zeroth-order substitution of (12) into (13) gives $(\mu-1)^2=0$ for all values of the phase velocity $c$. Therefore, corresponding characteristic exponents are $\gamma = \lambda^{-1}\log(\mu) = 0$. We now solve (9) for $\phi_1$ in terms of $\phi_0$, giving

$$\phi_1 = (U-c)\int(U-c)^{-2}\int^z (U-c)\,\phi_0\,dz_1\,dz+E(U-c)\int^z (U-c)^{-2}dz_1+F(U-c), \tag{14}$$

for some constants $E$ and $F$. Proceeding as before, we then obtain values of $\phi_1$ at $z=\lambda$ for the two sets of initial conditions (11). This gives

$$\phi_1(\lambda) = \{U(0)-c\}\int_0^\lambda (U-c)^{-2}\int_0^z (U-c)\,\phi_0\,dz_1\,dz,$$

$$\phi'_1(\lambda) = U'(0)\int_0^\lambda (U-c)^{-2}\int_0^z (U-c)\,\phi_0\,dz_1\,dz+(U(0)-c)^{-1}\int_0^\lambda (U-c)\,\phi_0\,dz. \tag{15}$$

We then substitute the two values of $\phi_0$ and add the diagonal terms in the fundamental matrix to obtain the equation for $\mu$ as

$$\mu^2-\{\phi_{0a}(\lambda)+\phi'_{0b}(\lambda)+\alpha^2(\phi_{1a}(\lambda)+\phi'_{1b}(\lambda))\}\mu+1 = 0, \tag{16}$$

to order $\alpha^2$. Now we have

$$\begin{aligned}\phi_{1a}(\lambda)+\phi'_{1b}(\lambda) &= \int_0^\lambda (U-c)^2\int_0^z (U-c)^{-2}\,dz_1\,dz+\int_0^\lambda (U-c)^{-2}\int_0^z (U-c)^2\,dz_1\,dz\\ &= \left(\int_0^\lambda (U-c)^2\,dz\right)\left(\int_0^\lambda (U-c)^{-2}\,dz\right).\end{aligned}$$

on integration by parts. Therefore (16) becomes

$$\mu^2-\left\{2+\alpha^2\left(\int_0^\lambda (U-c)^2\,dz\right)\left(\int_0^\lambda (U-c)^{-2}\,dz\right)\right\}\mu+1 = 0,$$

which on letting $\mu = e^{i\gamma_I\lambda}$, yields

$$\cos(\lambda\gamma_I) = 1+\tfrac{1}{2}\alpha^2\left(\int_0^\lambda (U-c)^2\,dz\right)\left(\int_0^\lambda (U-c)^{-2}\,dz\right)+0(\alpha^4) \quad \text{as} \quad \alpha\to 0. \tag{17}$$

Now, upon letting $\gamma_I = a\alpha+O(\alpha^2)$ as $\alpha\to 0$ for some fixed $a$, we find that equating terms of $O(\alpha^2)$ gives

$$\left(\int_0^\lambda (U-c)^2\,dz\right)\left(\int_0^\lambda (U-c)^{-2}\,dz\right) = -\lambda^2a^2. \tag{18}$$

Equation (18) yields an expression for the phase velocity $c$ in terms of the parameter $a=\gamma_I/\alpha$ for small values of the wavenumber $\alpha$. Examples of the use of this long-wave expression will be given in the next two sections and comparisons are made with analytic and numerical results. We note at this point that if $\gamma_I = 0$ then (17) reduces to

$$\left(\int_0^\lambda (U-c)^2\,dz\right)\left(\int_0^\lambda (U-c)^{-2}\,dz\right) = 0. \tag{19}$$

The values of the phase velocity which make the first integral zero are always complex (unless $U$ = constant, $-\infty < z < \infty$) whereas the second integral is bounded for complex values of $c$. In passing, we also note that the second integral is singular if $c$ is real and between the maximum and minimum values of $U$. Thus, all non-constant periodic flows are unstable to long-waves with phase velocity

$$c = \left\{\int_0^\lambda U\,dz \pm i\left(\int_0^\lambda U^2\,dz - \left(\int_0^\lambda U\,dz\right)^2\right)^{\frac{1}{2}}\right\}\Big/\lambda.$$

In theory it would be possible to develop similar expansions for the viscous problem but in practice the complexity of the algebra involved with a fourth-order system means that numerical computation would be necessary at some stage, thus defeating the object of the exercise.

## 4. Broken-line velocity profiles

As is the case for aperiodic flows (cf. Rayleigh 1945) it is possible to obtain exact solutions for various periodic broken-line velocity profiles. Although broken-line profiles only give good approximation to continuously varying profiles when the wave-number $\alpha$ is small they give rise to simple explicit solutions which illustrate the general method well. We have only considered an inviscid fluid because the algebra becomes very involved when viscosity is taken into account and because basic flows of a viscous fluid do not satisfy the equations of motion when the profile is not smooth. The conditions of continuity of pressure and normal velocity imply that at a discontinuity of $U$ or $U'$ we have

$$[(U-c)\phi' - U'\,\phi] = 0 \quad \text{and} \quad [\phi/(U-c)] = 0, \tag{20}$$

where the square brackets denote the difference across the discontinuity of their contents.

*(a) The triangular velocity profile*

Here we consider a flow in which $U(z)$ is given by

$$U(z) = \begin{cases} 2z/\pi, & 0 < z < \frac{1}{2}\pi \\ 2 - 2z/\pi, & \frac{1}{2}\pi < z < \frac{3}{2}\pi \\ 2z/\pi - 4, & \frac{3}{2}\pi < z < 2\pi \end{cases}$$

with $U(z+2\pi) = U(z)$.

We now use the two sets of initial conditions at $z_j = O((\phi, \phi')^T = (1, 0)^T$ and $(\phi, \phi')^T = (0, 1)^T$, denoted by subscripts $a$ and $b$ respectively) and proceed, by using the matching equations at $z = \frac{1}{2}\pi$ and $z = \frac{3}{2}\pi$, to solve the Rayleigh equation (2) at $z = 2\pi$. This at length gives

$$\left.\begin{aligned} \phi_a(2\pi) &= \cosh 2\pi\alpha - \frac{4(c\sinh\pi\alpha + \sinh 2\pi\alpha)}{\pi\alpha(1-c^2)} + \frac{4(\cosh 2\pi\alpha - 1)}{\pi^2\alpha^2(1-c^2)}, \\ \phi'_b(2\pi) &= \cosh 2\pi\alpha + \frac{4(c\sinh\pi\alpha - \sinh 2\pi\alpha)}{\pi\alpha(1-c^2)} + \frac{4(\cosh 2\pi\alpha - 1)}{\pi^2\alpha^2(1-c^2)}. \end{aligned}\right\} \tag{22}$$

The equation for the characteristic exponents $\mu$ is then

$$\mu^2 - \{\phi_a(2\pi) + \phi'_b(2\pi)\}\mu + 1 = 0,$$

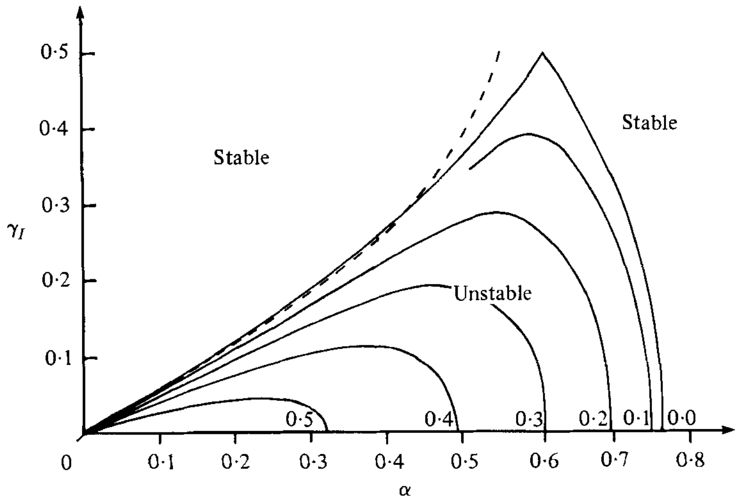


FIGURE 1. The stability curves (values of $c_I$ marked) for the triangular velocity profile. The dashed curve indicates the small-$\alpha$ approximation to the neutral curve. Note that $c(\gamma_I) = c(-\gamma_I)$ and $c(\gamma_I+1) = c(\gamma_I)$. Part of the $c_I = 0{\cdot}1$ curve has been omitted for the purpose of clarity.

which, on letting $\mu = \exp\{2\pi\gamma_I i\}$ and solving for $c^2$, gives

$$c^2 = \frac{\pi^2\alpha^2\cos 2\pi\gamma_I - \pi^2\alpha^2\cosh 2\pi\alpha - 4\cosh 2\pi\alpha + 4\alpha\pi\sinh 2\pi\alpha + 4}{\alpha^2\pi^2(\cos 2\pi\gamma_I - \cosh 2\pi\alpha)}. \tag{23}$$

Equation (23) agrees with the long-wave expression (18) for this profile, both these results giving $c \to (a^2 - 1/3)/(1+a^2)$ as $\alpha \to 0$, where $a = \gamma_I/\alpha$. The stability curves for this profile are shown in figure 1. Also as $\alpha \to \infty$, $c \to \pm 1$ and there are therefore stable short waves present on all the discontinuities of the profile (cf. triangular jet in Drazin & Howard 1966, p. 38). The dotted line gives the small-$\alpha$ approximation (equation (17)) for the neutral curve (that with $c_I = 0$). It should also be noted that the solution is invariant if 1 is added to $\gamma_I$ and that $c(-\gamma_I) = c(\gamma_I)$.

### (*b*) *The square velocity profile*

We now consider a flow in which $U(z)$ is given by:

$$U(z) = \begin{cases} 1, & 0 < z < \tfrac{1}{2}\pi \\ -1, & \tfrac{1}{2}\pi < z < \tfrac{3}{2}\pi \\ 1, & \tfrac{3}{2}\pi < z < 2\pi \end{cases}$$

with $U(z+2\pi) = U(z)$.

Following the same procedure as for the triangular wave velocity profile, we obtain a quartic equation for $c$:

$$c^4 + \frac{2(2-\cosh 2\pi\alpha - \cos 2\pi\gamma_I)}{(\cos 2\pi\gamma_I - \cosh 2\pi\alpha)}c^2 + 1 = 0. \tag{24}$$

Therefore

$$c_R = \pm\left\{\frac{1-\cos 2\pi\gamma_I}{\cosh 2\pi\alpha - \cos 2\pi\gamma_I}\right\}^{\frac{1}{2}}, \quad c_I = \pm\left\{\frac{\cosh 2\pi\alpha - 1}{\cosh 2\pi\alpha - \cos 2\pi\gamma_I}\right\}^{\frac{1}{2}}. \tag{25}$$

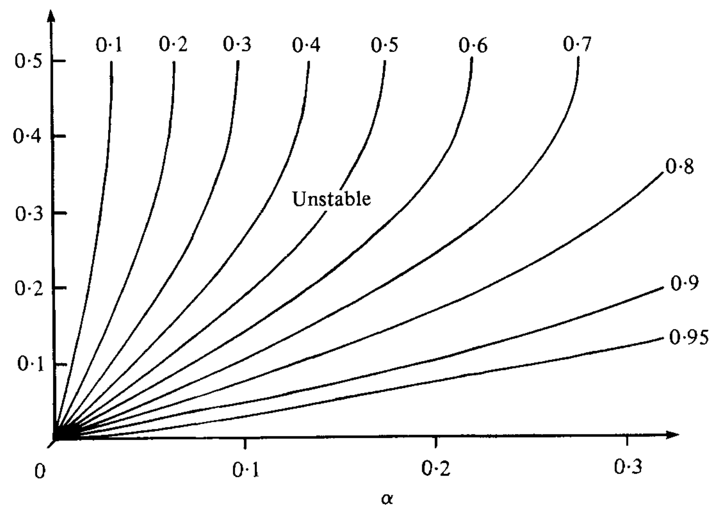


FIGURE 2. The stability curves (values of $c_I$ marked) for the rectangular velocity profile. Note that $c(\gamma_I) = c(-\gamma_I)$ and $c(\gamma_I+1) = c(\gamma_I)$. The whole of the $(\alpha,\ \gamma_I)$ plane is unstable.

Equations (25) have the following points of interest;

(i) $c \to \pm i$ as $\gamma_I \to 0$ for all $\alpha$;

(ii) $c_R^2 + c_I^2 = 1$ for all $\alpha$ and $\gamma_I$;

(iii) unstable waves of all lengths are possible;

(iv) $c \to \pm i$ as $\alpha \to \infty$, the eigenvalues for Helmholtz instability of a vortex sheet at each discontinuity. The small wavenumber expansion (18) for this profile gives $c \to \{\pm a \pm i\}/(1+a^2)^{\frac{1}{2}}$ as $\alpha \to 0$ where $a = \gamma_I/\alpha$ which agrees with equations (25). The stability curves are shown in figure 2 and once again the solution is invariant under the addition of 1 to $\gamma_I$ and $c(-\gamma_I) = c(\gamma_I)$, as before.

## 5. Inviscid stability of a sinusoidal velocity profile

We now consider the stability of the flow $U(z) = \sin z$, $-\infty < z < \infty$ in an inviscid fluid. The stability of this flow has been considered by Lorenz (1972) and Green (1974) but they only studied perturbations with the same period as the basic flow. The equation to be solved is therefore

$$(\sin z - c)(\phi'' - \alpha^2\phi) + \sin z\,\phi = 0. \tag{26}$$

First we shall consider solutions of (26) with $c = 0$. Consider solutions of the form $\phi = P(z)\exp(i\gamma_I z)$, where $P(z)$ has period $2\pi$. Then substitution into (26) gives

$$(\sin z - c)\{P'' + 2i\gamma_I P' - (\gamma_I^2 + \alpha^2)P\} + (\sin z)P = 0. \tag{27}$$

If $c = 0$ then it can be seen that (27) has a solution $P$ = constant (= 1, say) and $\gamma_I^2 + \alpha^2 = 1$. This solution is now perturbed by letting $\gamma_I{}^2 = 1 - \alpha^2 - \epsilon$, so that (27) becomes

$$P'' + 2i(1-\alpha^2-\epsilon)^{\frac{1}{2}}P' + \epsilon P/(\sin z - c).$$

Now, upon substitution for $P = 1 + \epsilon P_1 + O(\epsilon^2)$ and $c = \epsilon c_1 + O(\epsilon^2)$, it follows that

$$\epsilon P_1'' + 2\epsilon i(1-\alpha^2-\epsilon)^{\frac{1}{2}}P_1' + \epsilon = \frac{-\epsilon c_1 + O(\epsilon^2)}{(\sin z - \epsilon c_1 + O(\epsilon^2))} + O(\epsilon^2).$$

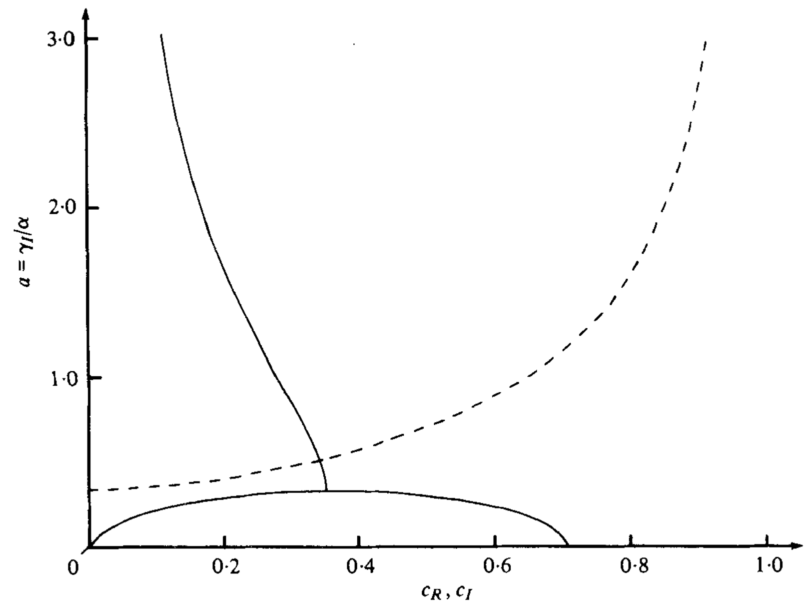


FIGURE 3. The valid unstable roots of the sextic equation for $c$ resulting from the use of the small $\alpha$ expansion for the sin $z$ velocity profile. The solid line gives the value of $c_I$ and the dashed line the corresponding value of $c_R$ at any given value of $a = \gamma_I/\alpha$. $c = c_R + ic_I \to 1$ as $a \to \infty$.

Thus, $$P_1'' + 2i(1-\alpha^2-\epsilon)^{\frac{1}{2}} P_1' = -1 - \frac{c_1 + O(\epsilon)}{(\sin z - \epsilon c_1 + O(\epsilon^2))} + O(\epsilon).$$

Therefore, integration over a period gives
$$O = -2\pi \pm \frac{2\pi i c_1}{(1+\epsilon^2 c_1^2 + O(\epsilon^2))^{\frac{1}{2}}} + O(\epsilon),$$

and so to zeroth in $\epsilon$ we find that $c_1 = \pm i$. Thus if we perturb the curve $\gamma_I^2 + \alpha^2 = 1$ by an amount $\epsilon$, the resulting perturbation in $c$ is $\pm \epsilon i$. It can be envisaged from figure 4 that this perturbation formula is a good approximation except near $\gamma_I = 0.5$, where it is clear that $c$ quickly becomes a non-linear function of $\epsilon$.

Before solving the eigenvalue problem (26) numerically, we consider the small-wavenumber approximation for this flow. Equation (18) yields the following equation for $c$ in terms of $a = \gamma_I/\alpha$:
$$(c^2-1)^{\frac{3}{2}} = (1+2c^2)\,c/2a^2.$$

On squaring (this creates some invalid roots), there results a cubic equation for $c^2$,
$$f(c^2) = c^6 + \{(1+3a^4)/(1-a^4)\}\,c^4 + \{(\tfrac{1}{4}-3a^4)/(1-a^4)\}\,c^2 + a^4/(1-a^4) = 0. \quad (28)$$

The discriminant of the cubic $f(c^2)$ is $\Delta = a^4(81a^4-1)/64(1-a^4)^4$ for all real $a$. Clearly $\Delta < 0$ if $a < \frac{1}{3}$ and $\Delta > 0$ if $a > \frac{1}{3}$. Therefore if $a < \frac{1}{3}$ all three roots $c^2$ of equation (28) are real and if $a > \frac{1}{3}$ two roots are complex conjugate and the third is real. The flow is unstable for all values of $a$ ($\alpha$ small) although $c \to 1$ as $a \to \infty$. Also, $c \to 0$ or $\pm i/\sqrt{2}$ as $a \to 0$. The valid unstable roots $c = c_R + ic_I$ of the sextic equation (28) are plotted in figure 3 and they were used as estimates in the computations of the 'exact' solution for $c_R \neq 0$.

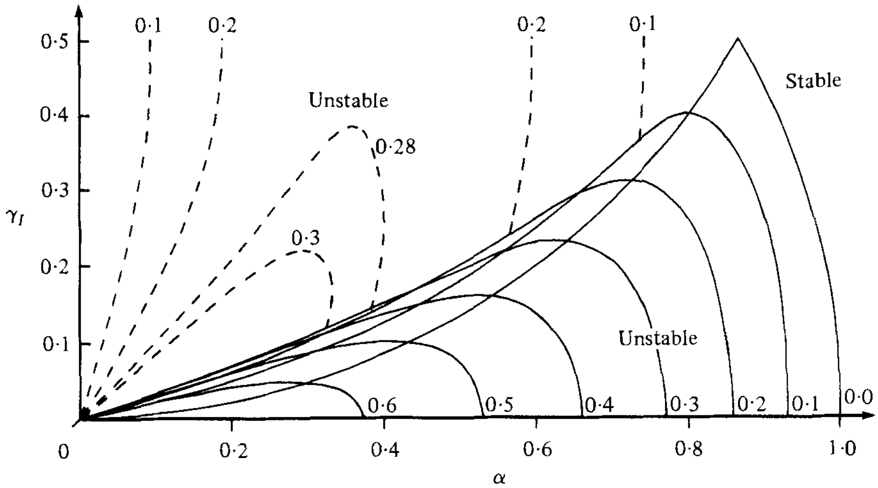


FIGURE 4. The stability curves (values of $c_I$ marked) for the $U(z) = \sin z$ profile in an inviscid fluid. The solid lines have $c_R = 0$ and the dashed lines have $c_R \neq 0$. Note that $c(-\gamma_I) = c(\gamma_I)$ and $c(\gamma_I + 1) = c(\gamma_I)$. For details of the behaviour near $\alpha = 0$ it is preferable to refer to figure 4.

Equation (26) was integrated from $z = 0$ to $z = 2\pi$ by use of a standard Runge–Kutta procedure, for fixed values of $c_I$ starting with the fundamental matrix

$$\mathbf{\Phi}(0) = \begin{bmatrix} \phi_1(0) & \phi_2(0) \\ \dot{\phi}_1(0) & \dot{\phi}_2(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{at} \quad z = 0.$$

The eigenvalues $\mu$ of the matrix $\mathbf{\Phi}(2\pi)$ were then found, and the corresponding characteristic multipliers $\gamma_I$ calculated. The stability curves on the range $0 \leqslant \gamma_I \leqslant \frac{1}{2}$ are shown in figure 4 where, once again, there is symmetry and periodicity in $\gamma_I$. All the curves with $\gamma_I/\alpha \leqslant \frac{1}{3}$ as $\alpha \to 0$ have $c_R = 0$. In order to obtain the stability curves with $c_R \neq 0$ iteration was used to vary the value of $c_R$ (or $\alpha$) whilst ensuring that $\gamma_R = 0$. There is a good check on the computations in that $\det \mathbf{\Phi}(2\pi) = 1$ and if this was not the case, to five places of decimals, the step length used in the integrations was decreased.

## 6. Viscous stability of a sinusoidally varying velocity profile

Following the problem for an inviscid fluid in the previous section we next look at the corresponding problem for a viscous fluid. In this section we shall expand the solution of the Orr–Sommerfeld equation in Fourier series and therefore, for ease of comparison with Green's (1974) solution with $\gamma_I = 0$, we shall transform the independent variable, letting $z \to z - \frac{1}{2}\pi$. This changes the eigenfunction $\phi$ but leaves the eigenvalues $c$ unaltered. We therefore consider solutions to

$$(\cos z - c)(\phi'' - \alpha^2\phi) + \cos z\phi = (i\alpha R)^{-1}\{\phi^{\text{iv}} - 2\alpha^2\phi'' + \alpha^4\phi\}$$

with $\phi$ given by
$$\phi = \sum_{n=-\infty}^{\infty} \frac{b_n e^{i(n+\gamma_I)z}}{\{(n+\gamma_I)^2 + \alpha^2 - 1\}}.$$

This yields the following recurrence relation for the coefficients $b_n$:

$$(b_{n+1} + b_{n-1})\left\{\frac{(n+\gamma_I)^2 + \alpha^2 - 1}{2\{(n+\gamma)^2 + \alpha^2\}}\right\} + b_n\left\{\frac{i}{\alpha R}((n+\gamma_I)^2 + \alpha^2) + c\right\} = 0, \quad n = 0, \pm 1, \ldots. \tag{29}$$

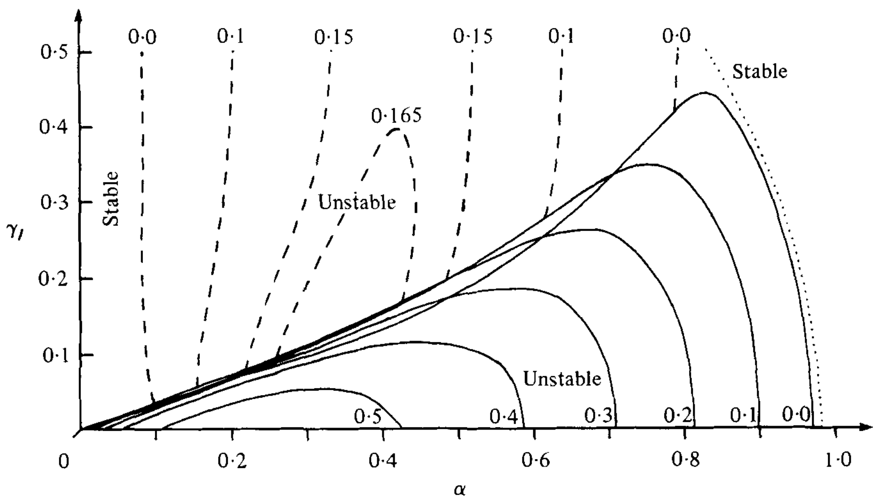


FIGURE 5. The stability curves (values of $c_I$ marked) for the $U(z) = \sin z$ profile in a viscous fluid with $R = 20$. The solid lines have $c_R = 0$ and the dashed lines have $c_R \neq 0$. The dotted line is the perturbation series approximation to the $c = 0$ curve. Note that $c(-\gamma_I) = c(\gamma_I)$ and $c(\gamma_I + 1) = c(\gamma_I)$.

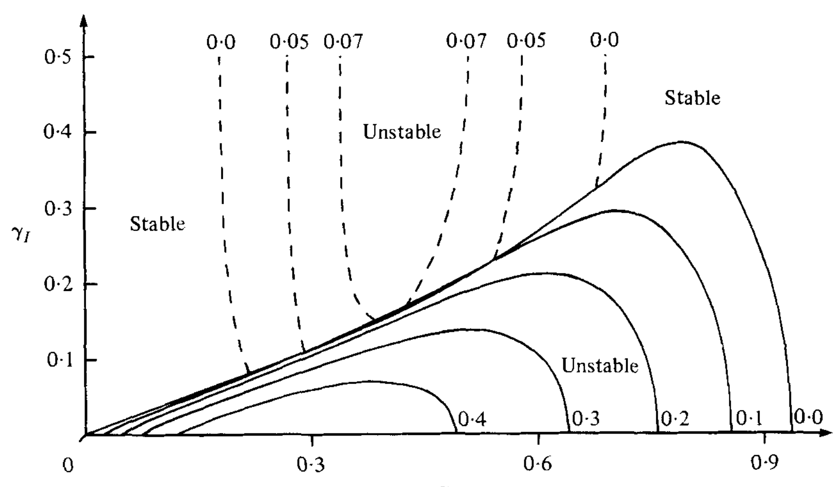


FIGURE 6. The stability curves (values of $c_I$ marked) for the $U(z) = \sin z$ profile in a viscous fluid with R = 10. The solid lines have $c_R = 0$ and the dashed lines have $c_R \neq 0$. Note that $c(-\gamma_I) = c(\gamma_I)$ and $c(\gamma_I + 1) = c(\gamma_I)$.

If we truncate the series at any given order the possible values of the phase speed $c$ are approximated by taking the negative of the eigenvalues of a tri-diagonal matrix defined by the relation (29) for given values of $\alpha$, $R$ and $\gamma_I$. The number of possible values of the phase speed will equal the order of the determinant under consideration. However, it is found that beyond the fifth order determinant any new eigenvalues which appear lead to stability of the basic flow. These eigenvalues appear to be similar to those found by Gotoh (1965) for the hyperbolic-tangent shear-layer but, because they give stable modes, we have not considered them in detail. We have therefore solved the problem by truncating the Fourier series at successively higher orders until the values of $c$ with $c_I > 0$ satisfy some criterion to indicate convergence. Interpolation is then used to obtain the relationship between $\gamma_I$ and $\alpha$ in order to remain on a $c_I =$ constant curve for a fixed value of $R$. The corresponding value of $c_R$ is also found by interpolation. It is found that the method always converges (to four places) is the eleventh-order determinant is considered (i.e. $b_{\pm 6} = 0$).

The direct integration method has also been used to solve the viscous problem but when modes with $c_R \neq 0$ are being considered (where iteration is necessary) for large $R$ the method becomes inefficient compared to the determinant method. It has therefore been used principally to check the determinant method at arbitrary points on the stability curves.

The stability curves for $R = 20$, 10, 7 and 4·5 are shown in figures 5–8 respectively. The stability curves for $R = 1{\cdot}7$ are similar in shape to those for $R = 4{\cdot}5$ but with a decrease by approximately a factor of ten in the maximum values of $c_I$ and $\gamma_I$. Green (1974) has pointed out that when $\gamma_I = 0$ the flow is stable for $R < \sqrt{2}$ and as the $c_R \neq 0$ curves disappear at a value of $R$ such that $6 < R < 7$ it seems that this condition will again hold (cf. figures 7 and 8). On all the dashed curves in figures 5–7 the value of $c_R$ is non-zero. It should also be noted that by a similar method to that used to perturb the $\gamma_I^2 + \alpha^2 = 1$ curve for the inviscid problem we can perturb this curve to find the $c = 0$ curve for large $\alpha R$. This gives $\gamma_I = (1-\alpha^2)^{\frac{1}{2}} - \frac{1}{2}(\alpha R)^{-1}(1-\alpha^2)^{-\frac{1}{2}} + O((\alpha R)^{-2})$ as $\alpha R \to \infty$ which is a good approximation to the $c = 0$ curve when $R = 20$ if $\alpha > 0{\cdot}9$ but, once again, as $\gamma_I$ approached 0.5 the behaviour of the stability curves is obviously non-linear. This approximation is shown as a dotted line in figure 5.

## 7. Discussion

In the preceding three sections we have examined the stability of some periodic parallel flows. Earlier workers (for a review see, e.g. Drazin & Howard 1966) have shown that broken-line profiles give useful approximations to the stability characteristics of continuously varying velocity profiles for small values of the wave-number $\alpha$. The inviscid stability characteristics of the sinusoidal velocity profile for small $\alpha$ are much more complex than those of either the triangular or square wave velocity profiles. However some of the qualitative characteristics of the sinusoidal velocity profile can be conjectured by the use of the results for broken-line profiles, i.e. for large values of $a = \gamma_I/\alpha$ the sinusoidal profile is similar to the square wave profile (cf. figures 2 and 4) and for small values of $\gamma_I/\alpha$ it is similar to the triangular velocity profile (cf. figures 1 and 4). In between there is a rather complicated structure which must be a combination of the two extreme broken-line approximations. It seems certain that a combination of the two broken-line approximations, e.g. a periodic trapezium profile, would provide a better approximation to the sinusoidal profile over a larger range of $a = \gamma_I/\alpha$ for small $\alpha$.

We have considered the stability of the sinusoidal velocity profile for both viscious and inviscid fluids. For the inviscid problem the neutral curve can be specified by $\alpha = 0$ and $\alpha = (1-\gamma_I^2)^{\frac{1}{2}}$ for $-\frac{1}{2} < \gamma_I < \frac{1}{2}$. The curves with $c_R \neq 0$ (dashed lines in figure 4) are of three types:

(i) curves emanating from the origin which reach the line $\gamma_I = 0{\cdot}5$ ($c_I < 0{\cdot}276$);

(ii) curves emanating from the origin which have a point where $d\gamma_I/d\alpha = 0$ and which then return to the corresponding $c_R = 0$ curve ($0{\cdot}276 < c_I < 0{\cdot}354$);

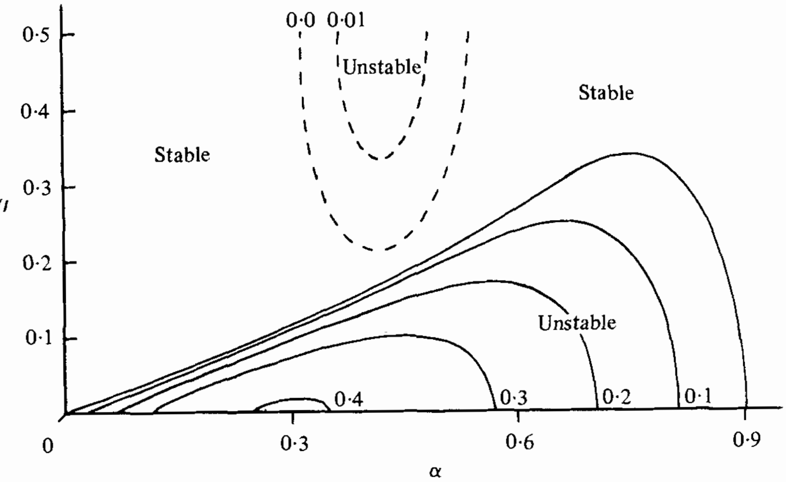


FIGURE 7. The stability (values of $c_I$ marked) for the $U(z) = \sin z$ profile in a viscous fluid with $R = 7$. The solid lines have $c_R = 0$ and the dashed lines have $c_R \neq 0$. All crossing of the $c_I = 0$ curves has now ceased. Note that $c(-\gamma_I) = c(\gamma_I)$ and $c(\gamma+1) = c(\gamma_I)$.

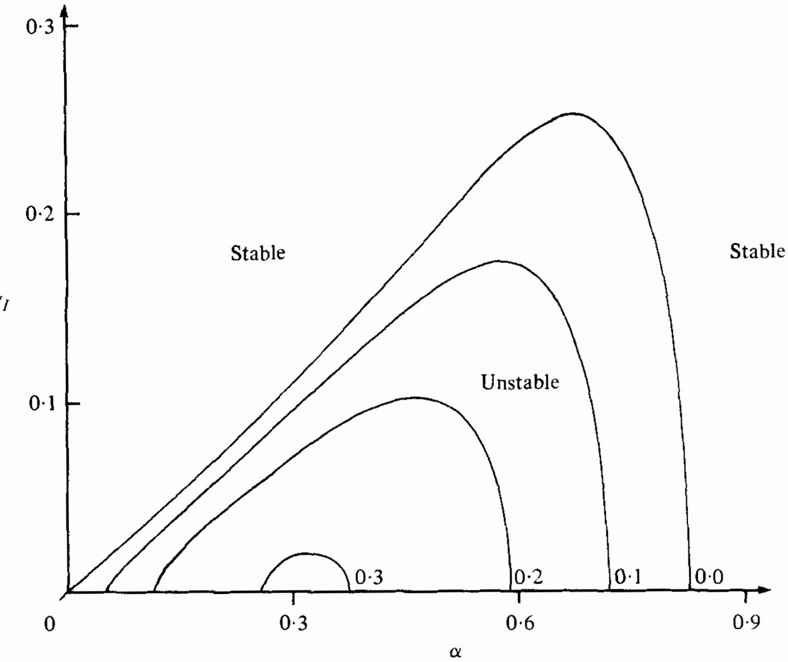


FIGURE 8. The stability curves (values of $c_I$ marked) for $U(z) = \sin z$ profile in a viscous fluid with $R = 4{\cdot}5$. $c_R = 0$ on all curves shown. Note that $c(-\gamma_I) = c(\gamma_I)$ and that $c(\gamma_I + 1) = c(\gamma_I)$.

(iii) curves emanating from the solid $c_R = 0$ curves which reach the line $\gamma_I = 0{\cdot}5$ ($c_I < 0{\cdot}276$).
The latter set of curves was found from those of type (ii) by iterating along a line of fixed $\gamma_I$.

The curves for $R = 20$ (figure 5) closely resemble those for $R = \infty$ (figure 4), the main difference being the movement of the $c_I \neq 0$, $c_R = 0$ curves away from the origin for $\gamma_I = 0$. There is also a general movement together of the dashed curves which have $c_R \neq 0$. In this case the minimum value of $c_I$ on the loops which have $c_R \neq 0$ is between 0·1635 and 0·1637. Between R = 20 and $R = 10$ a change takes place in the $c_R \neq 0$ curves and the maximum value of $c_I$ on these curves moves away from the $c_R = 0$ curves to the line $\gamma_I = 0{\cdot}5$ (figure 6). When $R = 7$ the curves with $c_R \neq 0$ have become separated from those with $c_R = 0$ and there is a stable region between the two sets of unstable curves (figure 7). This naturally means that the crossing of the $c = 0$ curve by other curves which have $c_R = 0$, $c_I > 0$ has stopped somewhere between $R = 7$ and $R = 10$. Further decrease of the Reynolds number to 6 leads to the disappearance of the $c_R \neq 0$ curves. When $R = 4{\cdot}5$ (figure 8) there is only one set of curves present (those with $c_R = 0$) and the flow is being stabilized as $R \to \sqrt{2}$.

Green (1974) remarked that it seems reasonable for the most unstable mode to have the same period as the basic flow (i.e. $\gamma_I = 0$). This is seen to be true for all the profiles considered in the present paper. However, although $c_I$ is an even function of $\gamma_I$, there seems no simple mathematical reason to suppose that $\gamma_I = 0$ always give the most unstable mode. It is also important to note that for the loops with $c_R \neq 0$ in the inviscid sin $(z)$ problem the most unstable wave does not have $\gamma_I = 0$ or $\gamma_I = 0{\cdot}5$ (see figure 4) although the semicircle theorem suggests that $c_R = 0$ gives the most unstable mode for this profile.

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